

The structure of strongly additive states and Markov triplets on the CAR algebra

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Abstract

We find a characterization of states satisfying equality in strong subadditivity of entropy and of Markov triplets on the CAR algebra. For even states, a more detailed structure of the density matrix is given.

1 Introduction

A remarkable property of von Neumann entropy is the strong subadditivity (SSA): For a state ρ on the 3-fold tensor product $B(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, we have

$$S(\rho) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$$

Here \mathcal{H}_A , \mathcal{H}_B and \mathcal{H}_C are finite dimensional Hilbert spaces and ρ_B , ρ_{AB} , ρ_{BC} are the restrictions of ρ to the respective subsystems. This was first proved by Lieb and Ruskai in [10].

The structure of states that saturate the strong subadditivity of entropy, called strongly additive states, was studied in [8]. It was shown that a state ρ is strongly additive if and only if it has the form

$$\rho = \bigoplus_n A_n \otimes B_n, \quad (1)$$

where $A_n \in B(\mathcal{H}_A \otimes \mathcal{H}_n)$ and $B_n \in B(\mathcal{K}_n \otimes \mathcal{H}_C)$ are positive operators and \mathcal{H}_B has a decomposition $\mathcal{H}_B = \bigoplus_n \mathcal{H}_n \otimes \mathcal{K}_n$ (see also [9], where this was proved also for the infinite dimensional case). Equivalently,

$$\rho = (D_{AB} \otimes I_C)(I_A \otimes D_{BC}) \quad (2)$$

where $D_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $D_{BC} \in B(\mathcal{H}_B \otimes \mathcal{H}_C)$ are positive matrices.

The Markov property for states in the quantum (non-commutative) probability was introduced by Accardi [1] and Accardi and Frigerio [3], in terms of completely positive unital maps, so-called quasiconditional

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expectations. For tensor products, it was shown that the Markov property is equivalent to strong additivity of the states [12].

The definition of the Markov property does not require the tensor product structure and can be applied in much more general situations. We are interested in the case of CAR algebras. The Markov states for CAR algebras were studied in [4]. The strong subadditivity of entropy on CAR systems was recently shown and it was proved that strong additivity is equivalent to Markov property in the case of even states, see [11]. For noneven states, a necessary and sufficient condition for equality in (SSA) was given in [6].

The aim of the present paper is to find the structure of strongly additive states and Markov triplets on the CAR algebra. We find an analogue of (2) for any states and of (1) for even states. This is done by a similar method as in [9], using the results of the theory of sufficient subalgebras.

The paper is organized as follows. The preliminary section summarizes the most important results on the CAR algebra and on sufficient subalgebras. The main tool used in the sequel is the factorization Theorem 2 in Section 2.1. Section 3 shows the relation between strong additivity and Markov property for any states on the CAR algebra. Section 4 contains the main results.

2 Preliminaries

2.1 Sufficient subalgebras

We first recall the definition and some characterizations of a sufficient subalgebra, which is a generalization of the classical notion of a sufficient statistic, see [13, 12] for details.

Let \mathcal{A} be a finite dimensional algebra and let φ, ψ be states on \mathcal{A} . Let $\mathcal{B} \subset \mathcal{A}$ be a subalgebra and let φ_0, ψ_0 be the restrictions of the states to \mathcal{B} . Then \mathcal{B} is sufficient for $\{\varphi, \psi\}$ if there is a completely positive, identity preserving map $E : \mathcal{A} \rightarrow \mathcal{B}$, such that $\varphi_0 \circ E = \varphi$, $\psi_0 \circ E = \psi$.

For simplicity, let us further assume that the states are faithful. Let ρ_φ, ρ_ψ be the densities of φ, ψ with respect to a trace Tr :

$$\varphi(a) = \text{Tr} \rho_\varphi a, \quad \psi(a) = \text{Tr} \rho_\psi a, \quad a \in \mathcal{A}$$

The relative entropy $S(\varphi, \psi)$ is defined as

$$S(\varphi, \psi) = S(\rho_\varphi, \rho_\psi) = \text{Tr} \rho_\varphi (\log \rho_\varphi - \log \rho_\psi)$$

It is monotone, in the sense that we have $S(\varphi, \psi) \geq S(\varphi_0, \psi_0)$ for any subalgebra $\mathcal{B} \subseteq \mathcal{A}$. We will also need the definition of the generalized conditional expectation $E_\psi : \mathcal{A} \rightarrow \mathcal{B}$ with respect to the state ψ [2]

$$E_\psi(a) = E_{\rho_\psi}(a) = \rho_{\psi_0}^{-1/2} E_{\mathcal{B}}(\rho_\psi^{1/2} a \rho_\psi^{1/2}) \rho_{\psi_0}^{-1/2}$$

where $E_{\mathcal{B}} : \mathcal{A} \rightarrow \mathcal{B}$ is the trace preserving conditional expectation. Then E_ψ is a completely positive identity preserving map, such that $\psi_0 \circ E_\psi = \psi$ and it is a conditional expectation if and only if $\rho_\psi^{it} \mathcal{B} \rho_\psi^{-it} \subseteq \mathcal{B}$ for all $t \in \mathbb{R}$.

The following theorem gives several equivalent characterizations of sufficiency.

Theorem 1. [12] *The following conditions are equivalent.*

- (i) *The subalgebra \mathcal{B} is sufficient for $\{\varphi, \psi\}$.*
- (ii) *$S(\varphi, \psi) = S(\varphi_0, \psi_0)$.*
- (iii) *$\rho_\varphi^{it} \rho_\psi^{-it} \in \mathcal{B}$, for all $t \in \mathbb{R}$.*
- (iv) *$E_\varphi = E_\psi$.*

Our results below are based on the following generalization of the classical factorization criterion for sufficient statistics.

Theorem 2. [9] *Let φ, ψ be faithful states on \mathcal{A} and let $\mathcal{B} \subseteq \mathcal{A}$ be a subalgebra, such that $\rho_\psi^{it} \mathcal{B} \rho_\psi^{-it} \subseteq \mathcal{B}$ for all $t \in \mathbb{R}$. Then \mathcal{B} is sufficient for $\{\varphi, \psi\}$ if and only if*

$$\rho_\varphi = \rho_{\varphi_0} D, \quad \rho_\psi = \rho_{\psi_0} D$$

where $\varphi_0 = \varphi|_{\mathcal{B}}$, $\psi_0 = \psi|_{\mathcal{B}}$ and D is a positive element in the relative commutant $\mathcal{B}' \cap \mathcal{A}$.

2.2 The CAR algebra

We recall some basic facts about the CAR algebra, for details see [5, 7].

The CAR algebra \mathcal{A} is the C^* -algebra generated by elements $\{a_i, i \in \mathbb{Z}\}$, satisfying the anticommutation relations

$$a_i a_j + a_j a_i = 0, \quad a_i a_j^* + a_j^* a_i = \delta_{ij}, \quad i, j \in \mathbb{Z} \quad (3)$$

For a subset $I \subset \mathbb{Z}$, the C^* -subalgebra generated by $\{a_i, i \in I\}$ is denoted by $\mathcal{A}(I)$. If I is finite, $\mathcal{A}(I)$ is isomorphic to the full matrix algebra $M_{2^{|I|}}(\mathbb{C})$ by the so-called Jordan-Wigner isomorphism. Since

$$\mathcal{A} = \overline{\bigcup_{|I| < \infty} \mathcal{A}(I)}^{C^*},$$

there is a unique tracial state τ on \mathcal{A} , obtained as an extension of the unique tracial states on $\mathcal{A}(I)$, $|I| < \infty$. It has the following product property:

$$\tau(ab) = \tau(a)\tau(b), \quad a \in \mathcal{A}(I), \quad b \in \mathcal{A}(J), \quad I \cap J = \emptyset \quad (4)$$

2.2.1 Graded commutation relations

For $I \subseteq \mathbb{Z}$, we denote by Θ^I the (unique) automorphism of \mathcal{A} , such that

$$\Theta^I(a_i) = -a_i, \quad i \in I, \quad \Theta^I(a_i) = a_i, \quad i \notin I \quad (5)$$

in particular, we denote $\Theta^{\mathbb{Z}}$ by Θ . The even and odd parts of \mathcal{A} are defined as

$$\mathcal{A}_+ := \{a \in \mathcal{A}, \Theta(a) = a\}, \quad \mathcal{A}_- := \{a \in \mathcal{A}, \Theta(a) = -a\}$$

and $\mathcal{A}(I)_+ := \mathcal{A}(I) \cap \mathcal{A}_+$, $\mathcal{A}(I)_- := \mathcal{A}(I) \cap \mathcal{A}_-$. Let $I \cap J = \emptyset$ and $a \in \mathcal{A}(I)_\sigma$, $b \in \mathcal{A}(J)_{\sigma'}$, $\sigma, \sigma' \in \{+, -\}$. Then we have the graded commutation relations

$$ab = \epsilon(\sigma, \sigma') ba \quad (6)$$

where

$$\begin{aligned}\epsilon(\sigma, \sigma') &= -1 & \text{if } \sigma = \sigma' = - \\ &= +1 & \text{otherwise}\end{aligned}$$

If I is finite, then there is a self-adjoint unitary $v_I \in \mathcal{A}(I)$, such that $\Theta^I(a) = v_I a v_I$ for $a \in \mathcal{A}$ and

$$v_I = \prod_{i \in I} v_i, \quad v_i = a_i^* a_i - a_i a_i^* \quad (7)$$

Note that $v_i v_j = v_j v_i$ if $i \neq j$ and $\tau(v_i) = 0$. Moreover, $v_I \in \mathcal{A}(I)_+$ and $\mathcal{A}(I)_+ = \mathcal{A}_I \cap \{v_I\}'$.

2.2.2 Matrix units

Let $A \subset \mathbb{Z}$ be a finite set, $A = \{i_1, \dots, i_n\}$. The relations

$$\begin{aligned}e_{11}^{(i_j)} &:= a_{i_j} a_{i_j}^*, & e_{12}^{(i_j)} &:= V_{i_{j-1}} a_{i_j} \\ e_{21}^{(i_j)} &:= V_{i_{j-1}} a_{i_j}^*, & e_{22}^{(i_j)} &:= a_{i_j}^* a_{i_j}\end{aligned}$$

with $V_{i_j} = \prod_{k=1}^j (I - 2a_{i_k}^* a_{i_k})$ define a family of mutually commuting 2×2 matrix units. The Jordan-Wigner isomorphism is then given by

$$e_{k_1 l_1 \dots k_n l_n}^{(A)} := e_{k_1 l_1}^{(i_1)} \dots e_{k_n l_n}^{(i_n)} \mapsto e_{k_1 l_1} \otimes \dots \otimes e_{k_n l_n}$$

where e_{kl} are standard matrix units in $M_2(\mathbb{C})$. The elements $\{e_\alpha^{(a)}, \alpha \in \mathcal{J}(A) := (\{1, 2\} \times \{1, 2\})^n\}$ span $\mathcal{A}(A)$. Note that $e_\alpha^{(A)}$ are either even or odd, we denote the set of indices of the even resp. odd elements by $\mathcal{J}(A)_+$, resp. $\mathcal{J}(A)_-$. Moreover, the elements $p_\alpha^{(A)} := e_\alpha^{(A)} (e_\alpha^{(A)})^*$ and $q_\alpha^{(A)} := (e_\alpha^{(A)})^* e_\alpha^{(A)}$ are even projections in $\mathcal{A}(A)$ and

$$p_\alpha^{(A)} e_\beta^{(A)} q_\alpha^{(A)} = \delta_{\alpha, \beta} e_\alpha^{(A)}, \quad \alpha, \beta \in \mathcal{J}(A) \quad (8)$$

2.2.3 Conditional expectations

Let $I \subseteq \mathbb{Z}$ be any subset. Then there is a unique conditional expectation $E_I : \mathcal{A} \rightarrow \mathcal{A}(I)$, satisfying

$$\tau(ab) = \tau(E_I(a)b), \quad a \in \mathcal{A}, \quad b \in \mathcal{A}(I) \quad (9)$$

This implies that $\Theta E_I = E_I \Theta$. If $J \subseteq \mathbb{Z}$, then $E_I(a) \in \mathcal{A}(I \cap J)$ for $a \in \mathcal{A}(J)$ and $E_I E_J = E_J E_I = E_{I \cap J}$. Note also that the product property (4) implies that for $a \in \mathcal{A}(J)$ with $I \cap J = \emptyset$, $E_I(a) = \tau(a)$.

3 Strong additivity and Markov property

Let A, B, C be disjoint finite subsets in \mathbb{Z} . Let us denote $\mathcal{A} = \mathcal{A}_{ABC} = \mathcal{A}(A \cup B \cup C)$, $\mathcal{A}_{AB} = \mathcal{A}(A \cup B)$ etc. Let φ be a faithful state on \mathcal{A} and let ρ be its density, that is, $\varphi(x) = \text{Tr} \rho x$ for $x \in \mathcal{A}$.

Let φ_{AB} denote the restriction of φ to \mathcal{A}_{AB} , similarly φ_{BC} and φ_B . Then the density of φ_{AB} in \mathcal{A}_{AB} is

$$\rho_{AB} = E_{AB}(\rho),$$

where $E_{AB} = E_{A \cup B}$. As an element in \mathcal{A} , ρ_{AB} is the density of the state $\varphi \circ E_{AB}$.

3.1 Strong subadditivity of entropy

Let ρ be the density of the state φ . Let

$$S(\varphi) = -\text{Tr}(\rho(\log(\rho)))$$

be the von Neumann entropy of φ . The strong subadditivity for CAR algebras

$$S(\varphi) - S(\varphi_{AB}) - S(\varphi_{BC}) + S(\varphi_B) \leq 0 \quad (\text{SSA})$$

was proved in [11]. This inequality is equivalent with

$$S(\rho, \rho_{BC}) - S(\rho_{AB}, \rho_B) \geq 0.$$

Since $\rho_{AB} = E_{AB}(\rho)$, $\rho_B = E_{AB}(\rho_{BC})$ are restrictions of ρ and ρ_{BC} to \mathcal{A}_{AB} , this holds by monotonicity of the relative entropy. Theorem 1 (ii) then implies the following.

Theorem 3. *The equality in (SSA) is attained if and only if the subalgebra \mathcal{A}_{AB} is sufficient for $\{\varphi, \varphi \circ E_{BC}\}$.*

3.2 Markov triplets and strong additivity

The state φ is a Markov triplet if there exists a completely positive, identity preserving map $E : \mathcal{A} \rightarrow \mathcal{A}_{AB}$, such that

- (i) $E(xy) = xE(y)$, for all $x \in \mathcal{A}_A$ and $y \in \mathcal{A}$.
- (ii) $\varphi \circ E = \varphi$
- (iii) $E(\mathcal{A}_{BC}) \subseteq \mathcal{A}_B$

The map E is called a quasi-conditional expectation with respect to the triplet $\mathcal{A}_A \subset \mathcal{A}_{AB} \subset \mathcal{A}$. Let us now define the subalgebras $\mathcal{B} \subset \mathcal{C}$ in \mathcal{A}_{AB} by

$$\mathcal{C} = \{x \in \mathcal{A}_{AB}, \rho_{BC}^{it} x \rho_{BC}^{-it} \in \mathcal{A}_{AB}\}, \quad \mathcal{B} = \{y \in \mathcal{A}_B, \rho_{BC}^{it} y \rho_{BC}^{-it} \in \mathcal{A}_B\}$$

Note that \mathcal{C} is the fixed point subalgebra of the generalized conditional expectation $E_{\rho_{BC}} : \mathcal{A} \rightarrow \mathcal{A}_{AB}$ with respect to ρ_{BC} [2]. We also have $E_{BC}(\mathcal{C}) = \mathcal{B}$. Indeed, if $x = E_{BC}(y)$ for some $y \in \mathcal{C}$, then

$$\rho_{BC}^{it} x \rho_{BC}^{-it} = E_{BC}(\rho_{BC}^{it} y \rho_{BC}^{-it}) \in \mathcal{A}_B,$$

so that $E_{BC}(\mathcal{C}) \subseteq \mathcal{B}$, the converse inclusion is clear.

Theorem 4. *The state φ is a Markov triplet if and only if φ satisfies equality in (SSA) and $\mathcal{A}_A \subseteq \mathcal{C}$.*

Proof. Let φ be a Markov triplet and let E be the quasi-conditional expectation. Then E is a completely positive identity preserving map $\mathcal{A} \rightarrow \mathcal{A}_{AB}$ and $\varphi \circ E = \varphi$. Moreover, let $x \in \mathcal{A}_A$, $y \in \mathcal{A}_{BC}$, then

$$\varphi \circ E_{BC} \circ E(xy) = \varphi \circ E_{BC}(xE(y)) = \tau(x)\varphi(E(y)) = \tau(x)\varphi(y) = \varphi \circ E_{BC}(xy)$$

Since by the commutation relations (3) \mathcal{A} is spanned by elements of the form xy , the above equality implies that E preserves $\varphi \circ E_{BC}$ as well, so that \mathcal{A}_{AB} is sufficient for $\{\varphi, \varphi \circ E_{BC}\}$ and equality in (SSA) holds by Theorem 3. Let

$$F = \lim_n \frac{1}{n} \sum_{k=0}^{n-1} E^k$$

By the ergodic theorem, F is a conditional expectation with range $\mathcal{R}(F)$ the fixed point subalgebra of E . By the property (i) of Markov triplets, $\mathcal{A}_A \subseteq \mathcal{R}(F)$. Since F also preserves $\varphi \circ E_{BC}$, we have by Takesaki theorem that $\rho_{BC}^{it} \mathcal{R}(F) \rho_{BC}^{-it} \subseteq \mathcal{R}(F)$, hence also $\rho_{BC}^{it} \mathcal{A}_A \rho_{BC}^{-it} \subseteq \mathcal{R}(F) \subseteq \mathcal{A}_{AB}$. It follows that $\mathcal{A}_A \subseteq \mathcal{C}$.

Conversely, suppose equality in (SSA) and $\mathcal{A}_A \subseteq \mathcal{C}$. Let $E = E_{\rho_{BC}} : \mathcal{A} \rightarrow \mathcal{A}_{AB}$ be the generalized conditional expectation. By Theorem 3, \mathcal{A}_{AB} is sufficient for $\{\varphi, \varphi \circ E_{BC}\}$ and by Theorem 1 (iv), $E_{\rho_{BC}} = E_\rho$, hence $\varphi \circ E = \varphi$. By the assumptions, $\mathcal{A}_A \subseteq \mathcal{C}$ the fixed point subalgebra of E . The property (iii) of Markov triplets is clear from the definition of $E_{\rho_{BC}}$. □

The following Corollary was already proved in [11].

Corollary 1. *Let φ be an even state. Then φ is a Markov triplet if and only if it satisfies equality in (SSA).*

Proof. Since ρ is even, ρ_{BC} is even as well and we always have $\mathcal{A}_A \subseteq \mathcal{C}$, by the graded commutation relations. The proof now follows from Theorem 4. □

4 Characterization of strongly additive states and Markov triplets

Theorem 5. *The state φ satisfies equality in (SSA) if and only if there are positive elements $x \in \mathcal{A}_{AB}$, $y \in \mathcal{A}_{BC}$, such that*

$$\rho = xy \tag{10}$$

Proof. Suppose that φ satisfies equality in (SSA). Then \mathcal{A}_{AB} is a sufficient subalgebra for $\{\varphi, \varphi \circ E_{BC}\}$. By Theorem 1, this implies that $u_t := \rho^{it} \rho_{BC}^{-it} \in \mathcal{A}_{AB}$ for all t . Since $\rho_{BC}^{it} u_s \rho_{BC}^{-it} = u_t^* u_{s+t}$ for $s, t \in \mathbb{R}$, this implies that $u_t \in \mathcal{C}$ for all t . Hence, \mathcal{C} is a sufficient subalgebra as well, such that $\rho_{BC}^{it} \mathcal{C} \rho_{BC}^{-it} \subseteq \mathcal{C}$. By Theorem 2,

$$\begin{aligned} \rho &= xy \\ \rho_{BC} &= x_0 y \end{aligned}$$

where $x, x_0 \in \mathcal{C} \subseteq \mathcal{A}_{AB}$ are the densities of the restrictions $\varphi|_{\mathcal{C}}$ and $\varphi \circ E_{BC}|_{\mathcal{C}}$ and y is a positive element in \mathcal{C}' . Note also that $\varphi \circ E_{BC}|_{\mathcal{C}}$ is the restriction of φ to $E_{BC}(\mathcal{C}) = \mathcal{B} \subseteq \mathcal{A}_B$, so that $x_0 \in \mathcal{A}_B$.

By the graded commutation relations, we have $(\mathcal{A}_A)_+ \subseteq \mathcal{C}$, so that $\mathcal{C}' \subseteq ((\mathcal{A}_A)_+)' = \mathcal{A}_{BC} + v_A \mathcal{A}_{BC}$, [5] (all commutants are taken in the algebra \mathcal{A}). Let $y \in \mathcal{C}'$, then $y = d_1 + v_A d_2$, where $d_1, d_2 \in \mathcal{A}_{BC}$. We have

$$x_0 y = E_{BC}(x_0 y) = E_{BC}(x_0(d_1 + v_A d_2)) = x_0 d_1.$$

Since φ , and therefore also its restriction to \mathcal{B} is faithful, x_0 is invertible, so that $y = d_1 \in \mathcal{A}_{BC}$.

Conversely, suppose $\rho = xy$ as above. Then $\rho_{AB} = xy_0$, $\rho_{BC} = x_0 y$ and $\rho_B = x_0 y_0$, where $y_0 = E_{AB}(y) \in \mathcal{A}_B$, $x_0 = E_{BC}(x) \in \mathcal{A}_B$. Clearly, both x and x_0 must commute with both y and y_0 . Then $\rho^{it} \rho_{BC}^{-it} = x^{it} x_0^{-it} \in \mathcal{A}_{AB}$. By Theorem 1 (iii), \mathcal{A}_{AB} is sufficient for $\{\varphi, \varphi \circ E_{BC}\}$, so that φ satisfies equality in (SSA). \square

Theorem 6. *The state φ is a Markov triplet if and only if there are positive elements $x \in \mathcal{A}_{AB}$ and $y \in (\mathcal{A}_{BC})_+$, such that*

$$\rho = xy$$

Proof. Let φ be a Markov triplet. By Theorem 4, φ satisfies equality in (SSA) and by Theorem 5 and its proof, there are positive elements $x \in \mathcal{C}$, $y \in \mathcal{C}'$, such that $\rho = xy$. Since $\mathcal{A}_A \subseteq \mathcal{C}$, $\mathcal{C}' \subseteq \mathcal{A}'_A = (\mathcal{A}_{BC})_+ + v_A (\mathcal{A}_{BC})_-$, [5]. This implies that $y = d_+ + v_A d_-$, where $d_+ \in (\mathcal{A}_{BC})_+$ and $d_- \in (\mathcal{A}_{BC})_-$. By the same reasoning as in the proof of Theorem 5, we get that $y = d_+ \in (\mathcal{A}_{BC})_+$.

Conversely, let $\rho = xy$ as above, then φ satisfies equality in (SSA) by Theorem 5, and $\rho_{BC} = x_0 y$, $x_0 = E_{BC}(x)$. For $a \in \mathcal{A}_A$,

$$\rho_{BC}^{it} a \rho_{BC}^{-it} = x_0^{it} a x_0^{-it} \in \mathcal{A}_{AB}$$

by the graded commutation relations, so that $\mathcal{A}_A \subseteq \mathcal{C}$. By Theorem 4, φ is a Markov triplet. \square

4.1 Even Markov triplets

Theorem 7. *Let φ be an even state. Then φ is a Markov triplet if and only if there are positive elements $x \in \mathcal{A}_{AB}$ and $y \in \mathcal{A}_{BC}$, such that*

$$\rho = xy.$$

Moreover, x and y can be chosen even.

Proof. Follows easily from Corollary 1, Theorems 5 and 6 and the fact that ρ is even. \square

We will now describe the subalgebras \mathcal{C} and \mathcal{C}' for even states. Since ρ_{BC} is even, both \mathcal{C} and \mathcal{B} and their commutants \mathcal{C}' and \mathcal{B}' are invariant under Θ .

Lemma 1. *If φ is even, then*

$$\mathcal{C} = \mathcal{A}_A \bigvee \mathcal{B}$$

Proof. Since $\mathcal{A}_A \subseteq \mathcal{C}$ and clearly also $\mathcal{B} \subseteq \mathcal{C}$, we have $\mathcal{A}_A \bigvee \mathcal{B} \subseteq \mathcal{C}$.

Conversely, any element $x \in \mathcal{C} \subseteq \mathcal{A}_{AB}$ has the form $x = \sum_{\alpha} e_{\alpha}^{(A)} b_{\alpha}$ for some $b_{\alpha} \in \mathcal{A}_B$, where $e_{\alpha}^{(A)}$ are the matrix units in \mathcal{A}_A . By (8), we have for any α ,

$$p_{\alpha}^{(A)} x q_{\alpha}^{(A)} = e_{\alpha}^{(A)} b_{\alpha},$$

since $q_{\alpha}^{(A)}$ is always even. As $\mathcal{A}_A \subseteq \mathcal{C}$, this implies that $e_{\alpha}^{(A)} b_{\alpha} \in \mathcal{C}$ for all α . It follows that

$$\rho_{BC}^{it} e_{\alpha}^{(A)} b_{\alpha} \rho_{BC}^{-it} = e_{\alpha}^{(A)} \rho_{BC}^{it} b_{\alpha} \rho_{BC}^{-it} \in \mathcal{A}_{AB},$$

hence $b_{\alpha} \in \mathcal{B}$, so that $\mathcal{C} \subseteq \mathcal{A}_A \bigvee \mathcal{B}$. □

Lemma 2. *If φ is even, then*

$$\mathcal{C}' = (\mathcal{B}' \cap \mathcal{A}_{BC})_+ + (\mathcal{B}' \cap \mathcal{A}_{BC})_- v_A$$

Proof. Since $\mathcal{A}_A \subseteq \mathcal{C}$, we have $\mathcal{C}' \subseteq \mathcal{A}'_A = (\mathcal{A}_{BC})_+ + (\mathcal{A}_{BC})_- v_A$, by [5]. Let $d_+ + v_A d_- \in \mathcal{C}'$ and let $x \in \mathcal{B} \subset \mathcal{C}$. Then we must have $x d_+ - d_+ x = v_A (d_- x - x d_-)$. Applying E_{BC} on both sides, we get $x d_+ - d_+ x = d_- x - x d_- = 0$, hence $d_+ \in \mathcal{B}' \cap (\mathcal{A}_{BC})_+ = (\mathcal{B}' \cap \mathcal{A}_{BC})_+$, $d_- \in \mathcal{B}' \cap (\mathcal{A}_{BC})_- = (\mathcal{B}' \cap \mathcal{A}_{BC})_-$.

Conversely, let $d_+ \in (\mathcal{B}' \cap \mathcal{A}_{BC})_+$, $d_- \in (\mathcal{B}' \cap \mathcal{A}_{BC})_-$ and let $a \in \mathcal{A}_A$, $b \in \mathcal{B}$. Then by the graded commutation relations,

$$ab(d_+ + v_A d_-) = d_+ ab + v_A d_- a b + v_A d_- a b = (d_+ + v_A d_-) ab$$

so that $d_+ + v_A d_- \in \mathcal{C}'$. □

Lemma 3. *Denote $\tilde{\mathcal{B}} = \mathcal{B}' \cap \mathcal{A}_B$. Then*

$$\mathcal{B}' \cap \mathcal{A}_{BC} = \tilde{\mathcal{B}} \bigvee ((\mathcal{A}_C)_+ + v_B (\mathcal{A}_C)_-)$$

Proof. It is easy to see that both $\tilde{\mathcal{B}}$ and $(\mathcal{A}_C)_+ + v_B (\mathcal{A}_C)_-$ are subsets in $\mathcal{B}' \cap \mathcal{A}_{BC}$. Conversely, any $y \in \mathcal{A}_{BC}$ has the form $y = \sum_{\beta} b_{\beta} e_{\beta}^{(C)}$, for some $b_{\beta} \in \mathcal{A}_B$. Let $x \in \mathcal{B}$, then

$$\begin{aligned} yx &= \sum_{\beta} b_{\beta} e_{\beta}^{(C)} (x_+ + x_-) = \sum_{\beta} b_{\beta} x_+ e_{\beta}^{(C)} + \sum_{\beta \in \mathcal{J}(C)_+} b_{\beta} x_- e_{\beta}^{(C)} - \sum_{\beta \in \mathcal{J}(C)_-} b_{\beta} x_- e_{\beta}^{(C)} \\ &= \sum_{\beta \in \mathcal{J}(C)_+} b_{\beta} x e_{\beta}^{(C)} + \sum_{\beta \in \mathcal{J}(C)_-} b_{\beta} \Theta(x) e_{\beta}^{(C)} \end{aligned}$$

It follows that $yx = xy$ only if $xb_{\beta} = b_{\beta}x$ for $\beta \in \mathcal{J}(C)_+$ and $xb_{\beta} = b_{\beta}\Theta(x)$ for $\beta \in \mathcal{J}(C)_-$. This is true for all $x \in \mathcal{B}$ if and only if $b_{\beta} \in \mathcal{B}'$ for $\beta \in \mathcal{J}(C)_+$ and $b_{\beta} v_B \in \mathcal{B}'$ for $\beta \in \mathcal{J}(C)_-$, this implies the statement of the lemma. □

Let us now look at the algebra \mathcal{B} . Let P_1, \dots, P_m be the minimal central projections in \mathcal{B} . Since \mathcal{B} is invariant under Θ , we must have for each i , $\Theta(P_i) = P_j$ for some j . Suppose that $\Theta(P_i) = P_i$, $i = 1, \dots, k$ and $\Theta(P_i) = P_{i+1}$ for $i = k + 2l + 1$, $l = 0, \dots, \frac{m-k}{2} - 1$.

Lemma 4. Let us denote $P_A = \frac{1}{2}(1 + v_A)$. The minimal central projections in \mathcal{C} are

$$\begin{aligned} Q_i &:= P_i, & i = 1, \dots, k \\ Q_{k+1} &:= P_A P_{k+1} + (1 - P_A) P_{k+2}, & Q_{k+2} := (1 - P_A) P_{k+1} + P_A P_{k+2} \\ &\dots \\ Q_{m-1} &:= P_A P_{m-1} + (1 - P_A) P_m, & Q_m := (1 - P_A) P_{m-1} + P_A P_m \end{aligned}$$

Proof. Clearly, $\mathcal{Z}(\mathcal{C}) \subset \mathcal{A}'_A \cap \mathcal{A}_{AB} = (\mathcal{A}_B)_+ + v_A(\mathcal{A}_B)_-$ and it is easy to see that if $x_+ + v_A x_- \in \mathcal{Z}(\mathcal{C})$, then x_+, x_- must be in $\mathcal{Z}(\mathcal{B})$. Therefore, $x_+ = \sum c_j P_j$ and $x_- = \sum_j d_j P_j$, for some $c_j, d_j \in \mathbb{C}$. Since x_+ is even, we must have $c_j = c_{j+1}$ for $j = k + 2l + 1, l = 0, \dots, \frac{m-k}{2} - 1$. Similarly, we get $d_j = 0$ for $j = 1, \dots, k$ and $d_j = -d_{j+1}$ for $j = k + 2l + 1, l = 0, \dots, \frac{m-k}{2} - 1$.

Suppose now that $P = x_+ + v_A x_-$ is a projection, then we must have $x_+^* x_+ + x_-^* x_- = x_+$ and $x_+^* x_- + x_-^* x_+ = x_-$. This implies that $c_j = |c_j|^2$ for $j = 1, \dots, k$, $c_j = |c_j|^2 + |d_j|^2 \geq 0$ and $c_j(d_j + \bar{d}_j) = d_j \in \mathbb{R}$, for $j > k$. Hence $2c_j d_j = d_j$, so that either $d_j = 0$ and then $c_j = c_j^2$, or $c_j = \frac{1}{2}$ and then $d_j = \pm \frac{1}{2}$.

It follows that any projection in $\mathcal{Z}(\mathcal{C})$ is a sum of some of the following projections: $P_i, i = 1, \dots, k, P_j + P_{j+1}, j = k + 2l + 1$, and $\frac{1}{2}(P_j + P_{j+1} \pm v_A(P_j - P_{j-1})), j = k + 2l + 1$. Since the last projection is equal to Q_j or Q_{j+1} and $Q_j + Q_{j+1} = P_j + P_{j+1}$, the Lemma follows. \square

Theorem 8. Let φ be an even faithful state on \mathcal{A} . Then φ is a Markov triplet if and only if there is an orthogonal family of projections $P_1, \dots, P_m \in \mathcal{A}_B$ and decompositions $P_j \mathcal{A}_B P_j = \mathcal{B}_j \otimes \tilde{\mathcal{B}}_j$, where \mathcal{B}_j and $\tilde{\mathcal{B}}_j$ are full matrix algebras, such that

1. $\Theta(P_j) = P_j$ and \mathcal{B}_j and $\tilde{\mathcal{B}}_j$ are invariant under Θ for $j = 1, \dots, k$
2. $\Theta(P_j) = P_{j+1}$ and $\Theta(\mathcal{B}_j) = \mathcal{B}_{j+1}, \Theta(\tilde{\mathcal{B}}_j) = \tilde{\mathcal{B}}_{j+1}$ for $j = k + 2l + 1, l = 0, \dots, \frac{m-k}{2} - 1$
3. Let us denote

$$\begin{aligned} V_j &= P_j v_B \\ \mathcal{C}_j &= \mathcal{A}_A \bigvee \mathcal{B}_j \\ \tilde{\mathcal{C}}_j &= \tilde{\mathcal{B}}_j \bigvee ((\mathcal{A}_C)_+ + V_j(\mathcal{A}_C)_-) \end{aligned}$$

for $j = 0, \dots, k$ and

$$\begin{aligned} U_l &= (P_{k+2l+1} + P_{k+2l+2}) v_B \\ \mathcal{D}_l &= \mathcal{A}_A \bigvee (P_A \mathcal{B}_{k+2l+1} + (1 - P_A) \mathcal{B}_{k+2l+2}) \\ \tilde{\mathcal{D}}_l &= (P_A \tilde{\mathcal{B}}_{k+2l+1} + (1 - P_A) \tilde{\mathcal{B}}_{k+2l+2}) \bigvee ((\mathcal{A}_C)_+ + U_l(\mathcal{A}_C)_-) \end{aligned}$$

for $l = 0, \dots, \frac{m-k}{2} - 1$, then there is a decomposition

$$\rho = \bigoplus_{j=1}^k x_j \otimes y_j \oplus \bigoplus_{l=0}^{\frac{m-k}{2}-1} (z_l \otimes w_l \oplus \Theta(z_l \otimes w_l)), \quad (11)$$

where $x_j \in \mathcal{C}_j$ and $y_j \in \tilde{\mathcal{C}}_j$ are positive and even for $j = 1, \dots, k$, and $z_l \in \mathcal{D}_l$, $w_l \in \tilde{\mathcal{D}}_l$ are positive for $l = 0, \dots, \frac{m-k}{2} - 1$.

Proof. Suppose that ρ has the form (11). Let us define Q_1, \dots, Q_m from P_1, \dots, P_m as in Lemma 4. Then Q_j are mutually orthogonal projections and it is easy to see that $Q_j x_j = x_j$, $Q_j y_j = y_j$ and $Q_{k+2l+1} z_l = z_l$, $Q_{k+2l+1} w_l = w_l$, $Q_{k+2l+2} \Theta(z_l) = \Theta(z_l)$, $Q_{k+2l+2} \Theta(w_l) = \Theta(w_l)$. Put

$$x = \bigoplus_j x_j \oplus \bigoplus_l (z_l \oplus \Theta(z_l)), \quad y = \bigoplus_j y_j \oplus \bigoplus_l (w_l \oplus \Theta(w_l))$$

then $x \in \mathcal{A}_{AB}$ and $y \in \mathcal{A}_{BC}$ are positive even elements and $\rho = xy$. By Theorem 7, this implies that φ is a Markov triplet.

Conversely, suppose that φ is an even Markov triplet. Then we have seen that $\rho = xy$, where $x \in \mathcal{C}$ and $y \in \mathcal{C}'$ are positive and even. By Lemmas 1, 2 and 3, $x \in \mathcal{A}_A \vee \mathcal{B}$ and $y \in \tilde{\mathcal{C}} := \tilde{\mathcal{B}} \vee ((\mathcal{A}_C)_+ + v_B(\mathcal{A}_C)_-)$.

Let P_1, \dots, P_m be the minimal central projections in \mathcal{B} and let $\mathcal{B}_j := P_j \mathcal{B}$, $\tilde{\mathcal{B}}_j := P_j \tilde{\mathcal{B}}$. Then \mathcal{B}_j and $\tilde{\mathcal{B}}_j$ are full matrix algebras and $P_j \mathcal{A}_B P_j = \mathcal{B}_j \otimes \tilde{\mathcal{B}}_j$. Moreover, we may suppose that there is some $k \leq m$ such that 1. and 2. are fulfilled.

The minimal central projections Q_1, \dots, Q_m in \mathcal{C} are given by Lemma 4. Let us denote $\mathcal{C}_j = Q_j \mathcal{C}$, $\mathcal{C}'_j = Q_j \mathcal{C}'$. Then each \mathcal{C}_j , \mathcal{C}'_j is isomorphic to a full matrix algebra and we have a decomposition

$$\mathcal{C} = \bigoplus_j \mathcal{C}_j \otimes \tilde{I}_j, \quad \mathcal{C}' = \bigoplus_j I_j \otimes \mathcal{C}'_j$$

Since we are interested only in even elements in \mathcal{C}' , we take the algebra $\tilde{\mathcal{C}}_j := Q_j \tilde{\mathcal{C}} \subset \mathcal{C}'_j$. For $j = 1, \dots, k$, \mathcal{C}_j are invariant under Θ . For $l = 0, \dots, \frac{m-k}{2} - 1$, let us denote $\mathcal{D}_l := \mathcal{C}_{k+2l+1}$, $E_l := Q_{k+2l+1} + Q_{k+2l+2} = P_{k+2l+1} + P_{k+2l+2}$. Then E_l is an even projection, the algebra $E_l \mathcal{C} = \mathcal{D}_l \oplus \Theta(\mathcal{D}_l)$ is invariant under Θ and even elements in $E_l \mathcal{C}$ are of the form $x \oplus \Theta(x)$, for some $x \in \mathcal{D}_l$. Similar relation hold for $\tilde{\mathcal{C}}_j$ and $\tilde{\mathcal{D}}_l := \tilde{\mathcal{C}}_{k+2l+1}$.

Let us denote $x_j := Q_j x$, $y_j := Q_j y$ for $j = 1, \dots, k$ and $x_j := E_l x$, $y_j := E_l y$ for $j = k + 2l + 1$, $l = 0, \dots, \frac{m-k}{2} - 1$. Then all x_j , y_j are positive and even and

$$\rho = \bigoplus_{j=1}^k x_j \otimes y_j \oplus \bigoplus_{l=0}^{\frac{m-k}{2}-1} x_{k+2l+1} y_{k+2l+1}$$

Moreover, for $j = k + 2l + 1$ we must have $x_j = z_l \oplus \Theta(z_l)$ for some positive $z_l \in \mathcal{D}_l$ and similarly $y_j = w_l \oplus \Theta(w_l)$ for positive $w_l \in \tilde{\mathcal{D}}_l$, $l = 0, \dots, \frac{m-k}{2} - 1$. The rest of the proof now follows from Lemmas 1 and 3. □

References

- [1] L. Accardi, On noncommutative Markov property, *Funct. Anal. Appl.* **9** (1975), 1-8

- [2] L. Accardi, C. Cecchini, Conditional expectations in von Neumann algebras and a theorem of Takesaki, *J. Functional. Anal.* **45**(1982), 245-273.
- [3] L. Accardi and A. Frigerio, Markovian cocycles, *Math. Proc. R. Ir. Acad.* **83** (1983), 251-263
- [4] L. Accardi, F. Fidaleo, F. Mukhamedov, Markov states and chains on the CAR algebra, *Inf. Dimen. Anal. Quantum Probab., Rel. Top.* **10** (2007), 165-184
- [5] H. Araki, H. Moriya, Equilibrium statistical mechanics of fermion lattice systems, *Rev. Math. Phys.* **15** (2003), 93-198
- [6] J. Pitrik, V.P. Belavkin, Notes on the equality in SSA of entropy on CAR algebra, <http://arxiv.org/abs/math-ph/0602035>, (2006)
- [7] O. Bratelli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, Springer-Verlag, Heidelberg, 1981
- [8] P. Hayden, R. Jozsa, D. Petz, A. Winter, Structure of states which satisfy strong subadditivity of quantum entropy with equality, *Commun. Math. Phys.* **246** (2004), 359-374
- [9] A. Jenčová and D. Petz, Sufficiency in quantum statistical inference, *Commun. Math. Phys.* **263**(2006), 259-276.
- [10] E.H. Lieb and M.B. Ruskai, Proof of the strong subadditivity of quantum-mechanical entropy, *J. Math. Phys.* **14** (1973), 1938-1941
- [11] H. Moriya, Markov property and strong additivity of von Neumann entropy for graded systems, *J. Math. Phys.* **47** (2006), 033510
- [12] M. Ohya and D. Petz, *Quantum Entropy and Its Use*, Springer-Verlag, Heidelberg, 1993
- [13] D. Petz, Sufficiency of channels over von Neumann algebras, *Quart. J. Math. Oxford* **39** (1988), 97-108